Le transport optimal en pratique : géométrie, algorithmes et applications

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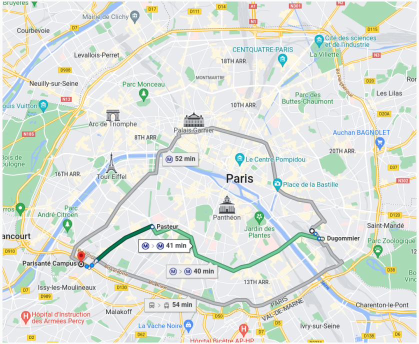
Mathematic Park, Institut Henri Poincaré, Paris

Who am I?

Background in **mathematics** and **data sciences**:

- **2007–2012** Highschool-MPSI-MP in the Lycée Marcelin Berthelot, Val-de-Marne.
- **2012–2016** ENS Paris, mathematics.
- **2014–2015** M2 mathematics, vision, learning at ENS Cachan.
- **2016–2019** PhD thesis in **medical imaging** with Alain Trouvé at ENS Cachan.
- **2019–2021** **Geometric deep learning** with Michael Bronstein at Imperial College.
- **2021+** **Medical data analysis** in the HeKA INRIA team (Paris).

Now working in PariSanté Campus, Porte de Versailles



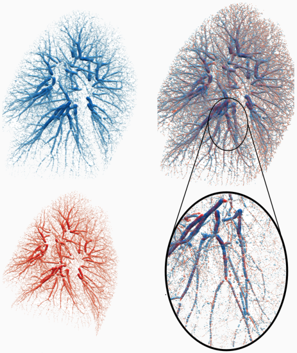
HeKA : a translational research team for public health

Hospitals

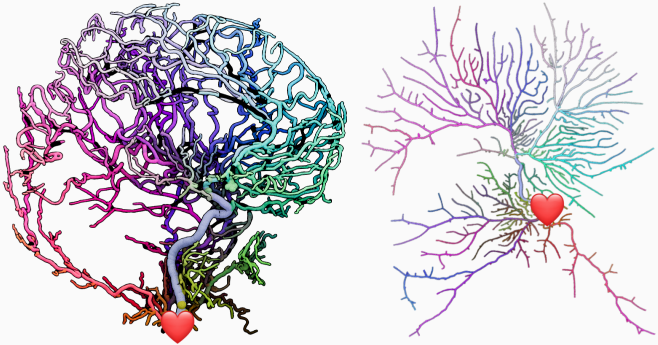
Inria Inserm

Universities

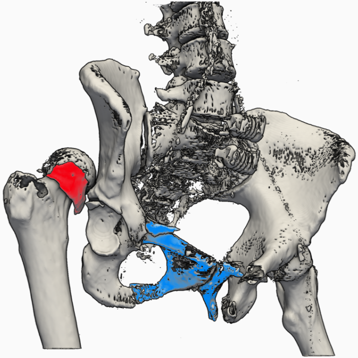
Recent works



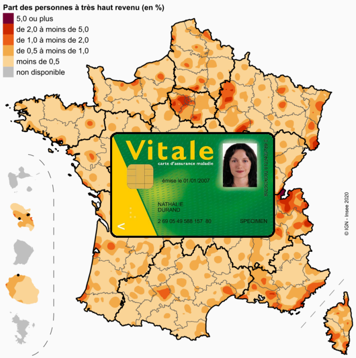
Lung **registration**.



Interventional **radiology**.

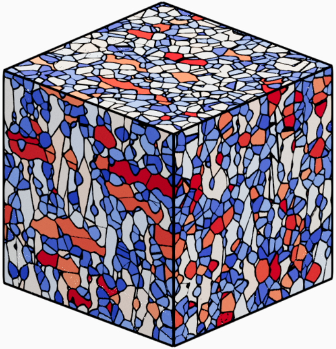


Orthopedic **surgery**.

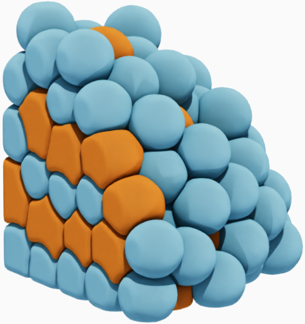


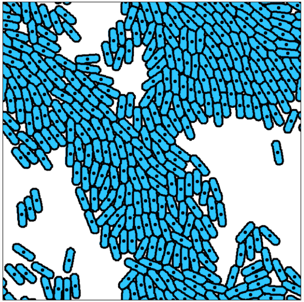
Public health.

Recent works



Metallurgy.





Swarms of incompressible **cells**.

My main motivation

Develop **robust and efficient** software that **stimulates other researchers**:

1. Speed up **geometric machine learning** on GPUs:
   $\Longrightarrow$ **pyKeOps** library for distance and kernel matrices, 700k+ downloads.
2. Scale up **pharmacovigilance** to the full French population:
   $\Longrightarrow$ **survivalGPU**, a fast re-implementation of the R survival package.
3. Ease access to modern statistical **shape analysis**:
   $\Longrightarrow$ **GeomLoss**, truly scalable optimal transport in Python.
   $\Longrightarrow$ **scikit-shapes**, alpha release now available.

Today’s talk – assuming that you would enjoy some nice simulations

1. The optimal transport **problem**.
2. Efficient discrete optimal transport **solvers**.
3. New applications for systems of **incompressible particles**.

Optimal transport?

Optimal transport (OT) generalizes sorting to spaces of dimension $D > 1$

If $A = (x_1, \dots, x_N)$ and $B = (y_1, \dots, y_N)$ are two clouds of N points in $\mathbb{R}^D$, we define:

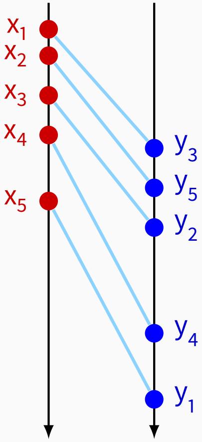
$$\text{OT}(A, B) = \min_{\sigma \in \mathcal{S}_N} \frac{1}{2N} \sum_{i=1}^{N} \| x_i - y_{\sigma(i)} \|^2$$

Generalizes **sorting** to metric spaces.

Linear problem on the permutation matrix π:

$$\text{OT}(A, B) = \min_{\pi \in \mathbb{R}^{N \times N}} \frac{1}{2N} \sum_{i,j=1}^{N} \pi_{i,j} \cdot \| x_i - y_j \|^2 ,$$

$$\text{s.t.} \quad \pi_{i,j} \geqslant 0 \qquad \underbrace{\textstyle\sum_j \pi_{i,j} = 1}_{\text{Each source point...}} \qquad \underbrace{\textstyle\sum_i \pi_{i,j} = 1}_{\text{is transported onto the target.}} .$$



assignment

$\sigma : [\![1, 5]\!] \rightarrow [\![1, 5]\!]$

Practical use

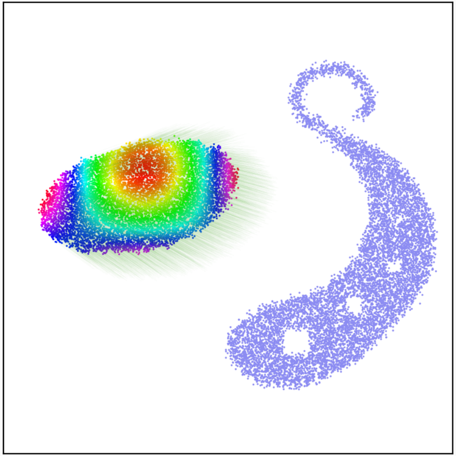
Alternatively, we understand OT as:

- Nearest neighbor **projection** + **incompressibility** constraint.
- Fundamental example of **linear optimization** over the transport plan $\pi_{i,j}$.

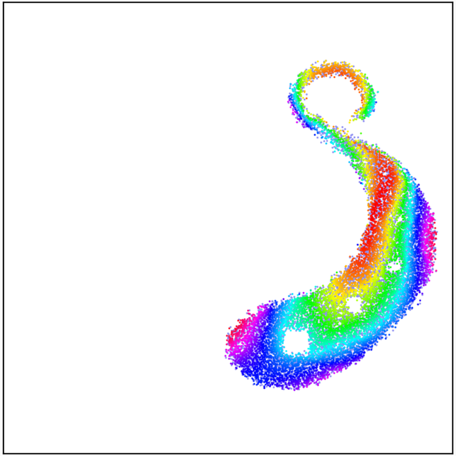
This theory induces two main quantities:

- The transport plan $\pi_{i,j} \simeq$ the optimal mapping $x_i \mapsto y_{\sigma(i)}$.
- The "Wasserstein" distance $\sqrt{\text{OT}(\text{A}, \text{B})}$.

The optimal transport plan

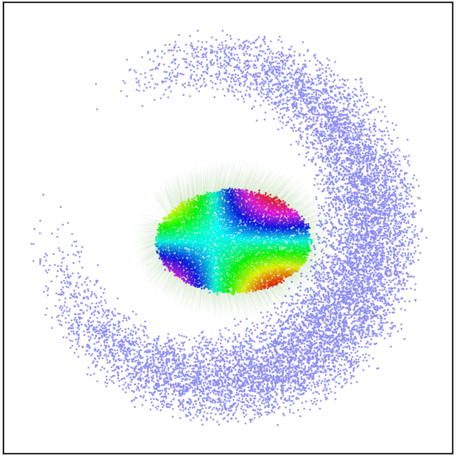


Before

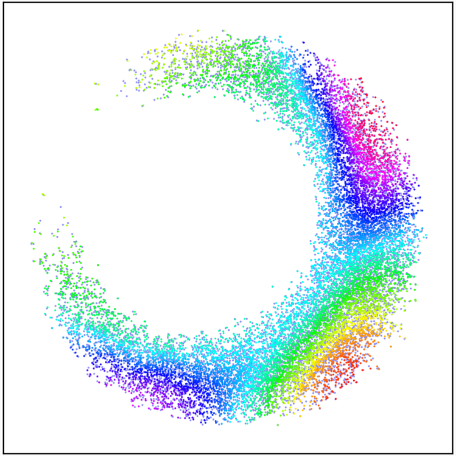


After

The optimal transport plan

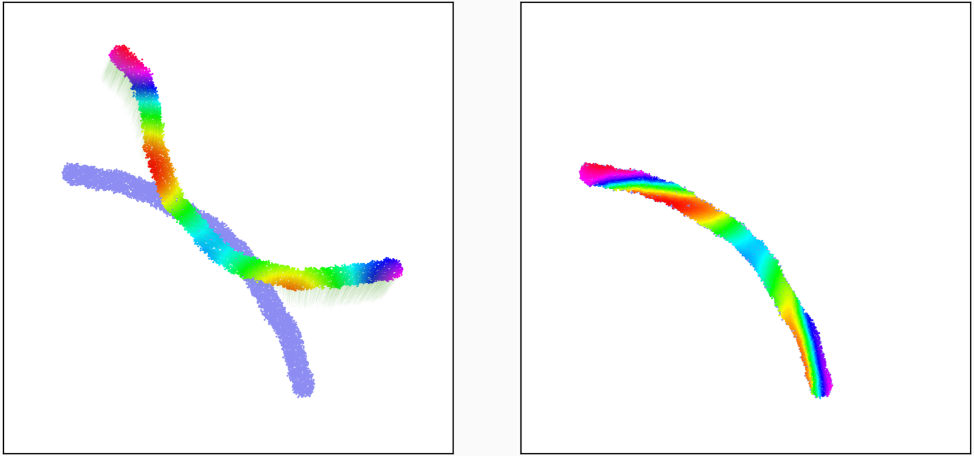


Before



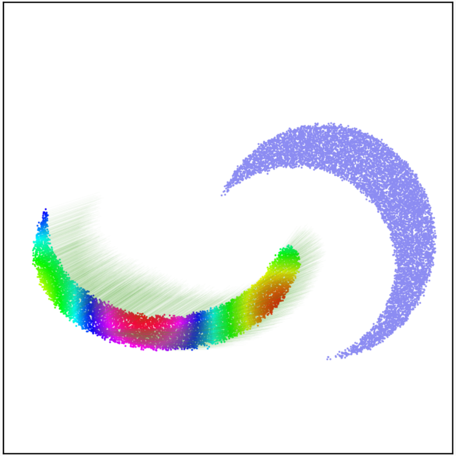
After

The optimal transport plan

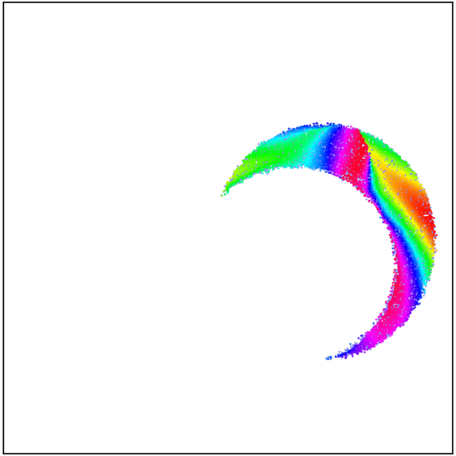


Before After

The optimal transport plan



Before

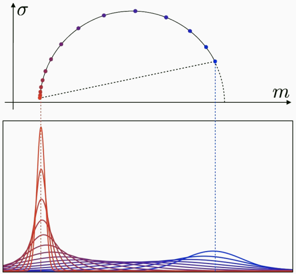


After

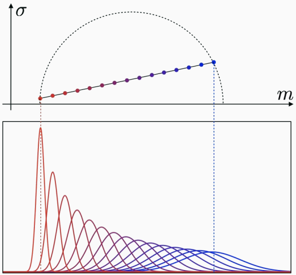
OT induces a geometry-aware distance between probability distributions [PC18]

Gauss map $\mathcal{N} : (m, \sigma) \in \mathbb{R} \times \mathbb{R}_{\geqslant 0} \;\mapsto\; \mathcal{N}(m, \sigma) \in \mathbb{P}(\mathbb{R})$.

If the space of **probability distributions** $\mathbb{P}(\mathbb{R})$ is endowed with a given metric, what is the "pull-back" geometry on the space of **parameters** (m, σ)?



Fisher-Rao ($\simeq$ relative entropy) on $\mathcal{N}(m, \sigma)$
$\rightarrow$ Hyperbolic **Poincaré** metric on (m, σ).



OT on $\mathcal{N}(m, \sigma)$
$\rightarrow$ Flat **Euclidean** metric on (m, σ).

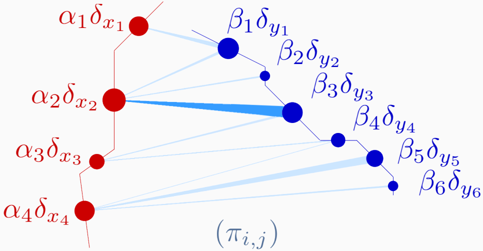
How to solve the OT problem?

Duality: central planning with NM variables $\simeq$ outsourcing with N + M variables

$$\text{OT}(A, B) = \min_{\pi} \langle \pi , C \rangle, \text{ with } C(x_i, y_j) = \tfrac{1}{p}\|x_i - y_j\|^p \quad \longrightarrow \text{ Assignment}$$

$$\text{s.t. } \pi \geqslant 0, \quad \pi \mathbf{1} = \alpha, \quad \pi^\top \mathbf{1} = \beta$$

$\sum_{i,j} \pi_{i,j} \, C(x_i, y_j)$

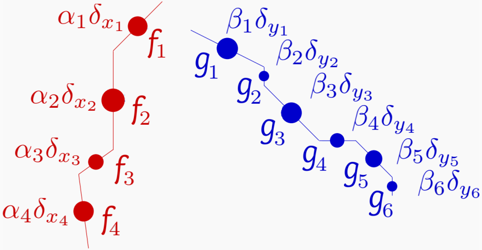


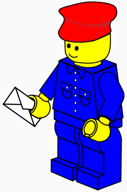
Duality: central planning with NM variables $\simeq$ outsourcing with N + M variables

$$\mathrm{OT}(\mathrm{A},\mathrm{B}) \;=\; \min_{\pi} \langle \pi\,,\,\mathrm{C}\rangle, \text{ with } \mathrm{C}(x_i,y_j) \;=\; \tfrac{1}{p}\|x_i - y_j\|^p \qquad \longrightarrow \text{ Assignment}$$

$$\text{s.t. } \pi \geqslant 0, \quad \pi\mathbf{1} = \alpha, \quad \pi^\top\mathbf{1} = \beta$$

$\sum_{i,j} \pi_{i,j}\,\mathrm{C}(x_i,y_j)$





$\sum_i \alpha_i f_i + \sum_j \beta_j g_j$

$\longrightarrow$ FedEx

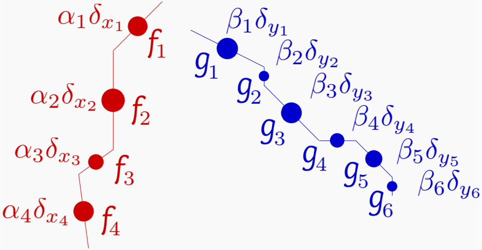
$$\max_{f\,,\,g} \quad \langle \alpha\,,\,f\rangle + \langle \beta\,,\,g\rangle$$

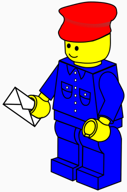
$$\text{s.t.} \quad f(x_i) + g(y_j) \leqslant \mathrm{C}(x_i,y_j),$$

Duality: central planning with NM variables $\simeq$ outsourcing with N + M variables

$$\text{OT}(\text{A}, \text{B}) = \min_{\pi} \langle \pi , \text{C} \rangle, \text{ with } \text{C}(x_i, y_j) = \tfrac{1}{p}\|x_i - y_j\|^p \quad \longrightarrow \text{ Assignment}$$

$$\text{s.t. } \pi \geqslant 0, \quad \pi \mathbf{1} = \alpha, \quad \pi^\top \mathbf{1} = \beta$$





$\sum_{i,j} \pi_{i,j} \text{C}(x_i, y_j)$

$\sum_i \alpha_i f_i + \sum_j \beta_j g_j$

$$= \max_{f, g} \langle \alpha , f \rangle + \langle \beta , g \rangle \quad \longrightarrow \text{ FedEx}$$

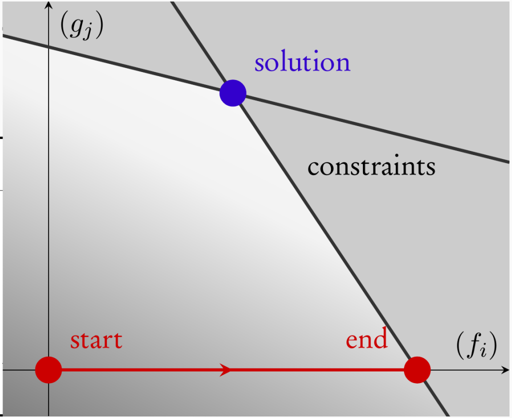
$$\text{s.t. } f(x_i) + g(y_j) \leqslant \text{C}(x_i, y_j),$$

Being too greedy... doesn't work!

$$\mathrm{OT}(\alpha,\beta) = \max_{\substack{(f_i)\in\mathbb{R}^{\mathrm{N}} \\ (g_j)\in\mathbb{R}^{\mathrm{M}}}} \sum_{i=1}^{\mathrm{N}} \alpha_i f_i + \sum_{j=1}^{\mathrm{M}} \beta_j g_j$$
$$\text{s.t. } \forall\, i,j,\ f_i + g_j \leqslant \mathbf{C}(x_i, y_j)$$

Algorithm 3.1: Naive greedy algorithm

1: $f_i,\ g_j \leftarrow \mathbf{0}_{\mathbb{R}^{\mathrm{N}}},\ \mathbf{0}_{\mathbb{R}^{\mathrm{M}}}$
2: **repeat**
3: $\quad f_i \leftarrow \min_{j=1}^{\mathrm{M}} \left[\, \mathbf{C}(x_i, y_j) - g_j \,\right]$
4: $\quad g_j \leftarrow \min_{i=1}^{\mathrm{N}} \left[\, \mathbf{C}(x_i, y_j) - f_i \,\right]$
5: **until** convergence.
6: **return** $f_i,\ g_j$



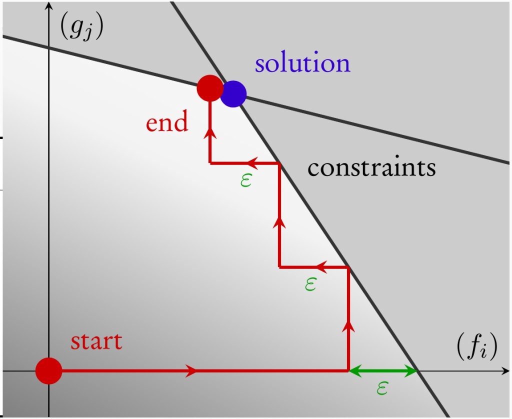
The auction algorithm: take it easy with a slackness $\varepsilon > 0$

$$\mathrm{OT}(\alpha, \beta) = \max_{\substack{(f_i) \in \mathbb{R}^N \\ (g_j) \in \mathbb{R}^M}} \sum_{i=1}^{N} \alpha_i f_i + \sum_{j=1}^{M} \beta_j g_j$$

$$\text{s.t. } \forall\, i, j,\; f_i + g_j \leqslant \mathbf{C}(x_i, y_j)$$

Algorithm 3.2: Pseudo-auction algorithm

1: $f_i,\; g_j \;\leftarrow\; \mathbf{0}_{\mathbb{R}^N},\; \mathbf{0}_{\mathbb{R}^M}$
2: **repeat**
3: $\quad f_i \;\leftarrow\; \min_{j=1}^{M} \left[\, \mathbf{C}(x_i, y_j) - g_j \,\right] - \varepsilon$
4: $\quad g_j \;\leftarrow\; \min_{i=1}^{N} \left[\, \mathbf{C}(x_i, y_j) - f_i \,\right]$
5: **until** $\forall\, i, \exists\, j,\; f_i + g_j \geqslant \mathbf{C}(x_i, y_j) - \varepsilon$.
6: **return** $f_i,\; g_j$

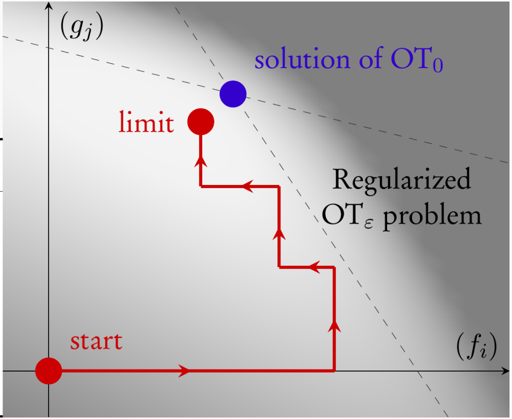


The Sinkhorn algorithm: use a softmin, get a well-defined optimum

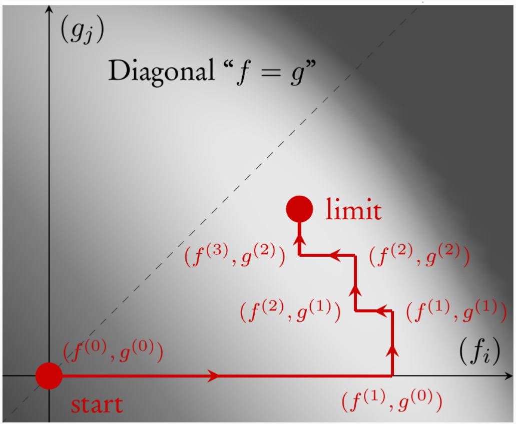
$$\mathrm{OT}(\alpha,\beta) = \max_{\substack{(f_i)\in\mathbb{R}^N \\ (g_j)\in\mathbb{R}^M}} \sum_{i=1}^{N} \alpha_i f_i + \sum_{j=1}^{M} \beta_j g_j - \varepsilon \log \langle \alpha_i \otimes \beta_j, \exp \tfrac{1}{\varepsilon}[f_i \oplus g_j - \mathbf{C}_{ij}] \rangle$$

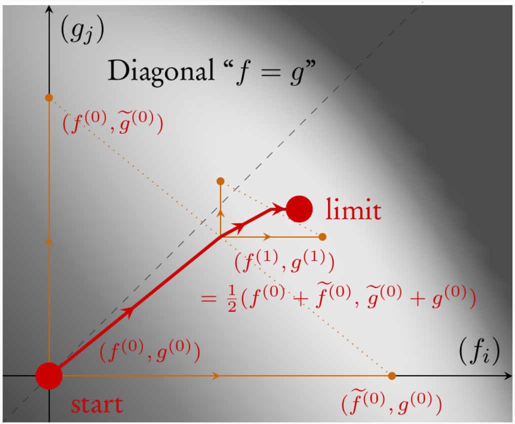
Algorithm 3.3: Sinkhorn or "soft-auction" algorithm

1: $f_i,\ g_j \leftarrow \mathbf{0}_{\mathbb{R}^N},\ \mathbf{0}_{\mathbb{R}^M}$
2: **repeat**
3: $\quad f_i \leftarrow -\varepsilon \log \sum_{j=1}^{M} \beta_j \exp \tfrac{1}{\varepsilon}[\, g_j - \mathbf{C}(x_i, y_j) \,]$
4: $\quad g_j \leftarrow -\varepsilon \log \sum_{i=1}^{N} \alpha_i \exp \tfrac{1}{\varepsilon}[\, f_i - \mathbf{C}(x_i, y_j) \,]$
5: **until** convergence up to a set tolerance.
6: **return** $f_i,\ g_j$



The symmetric Sinkhorn algorithm: stay close to the diagonal if A $\simeq$ B





Remark 1: a streamlined algorithm

One key operation – the soft, **weighted distance transform**:

$$\forall i \in [1, \mathsf{N}], \;\; f(x_i) \leftarrow \min_{y\sim\beta} \big[\mathsf{C}(x_i, y) - g(y)\big] = -\varepsilon \log \sum_{j=1}^{\mathsf{M}} \beta_j \exp \tfrac{1}{\varepsilon}\big[g_j - \mathsf{C}(x_i, y_j)\big] \,.$$

Similar to the chamfer distance transform, convolution with a Gaussian kernel...

Fast implementations with **pyKeOps**:

- If $\mathsf{C}(x_i, y_j)$ is a closed formula: **bruteforce** scales to N, M $\simeq$ 100k in 10ms on a GPU.
- If A and B have a low-dimensional support:
  use a clustering and **truncation** strategy to get a x10 speed-up.
- If A and B are supported on a 2D or 3D grid and $\mathsf{C}(x_i, y_j) = \frac{1}{2}\|x_i - y_j\|^2$:
  use a **separable** distance transform to get a second x10 speed-up.
  (N.B.: FFTs run into numerical accuracy issues.)

Remark 2: annealing works!

The **Auction/Sinkhorn** algorithms:

- Improve the dual cost by at least ε at each (early) step.
- Reach an ε-optimal solution with $(\max \text{C}) \,/\, \varepsilon$ steps.

Simple heuristic: run the optimization with **decreasing values** of ε.

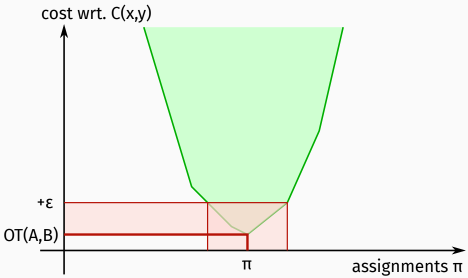
ε-scaling

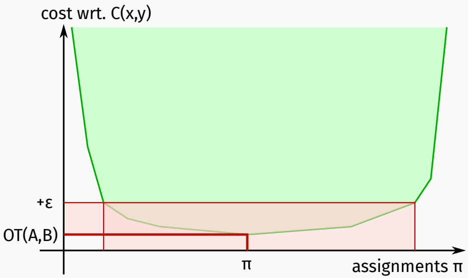
= **simulated annealing**

= **multiscale** strategy

= **divide and conquer**

Remark 3: the curse of dimensionality





In **low dimension**:

- $\|x - y\|$ takes large and small values.
- The OT objective is **peaky** wrt. π.
- ε-optimal solutions are **useful**.
- OT(discrete samples) $\simeq$ OT(underlying distributions)

In **high dimension**:

- $\|x - y\|$ gets closer to a constant.
- The OT objective is **flat** wrt. π.
- ε-optimal solutions are **random**.
- OT(discrete samples) $\neq$ OT(underlying distributions)

To recap 80+ years of work...

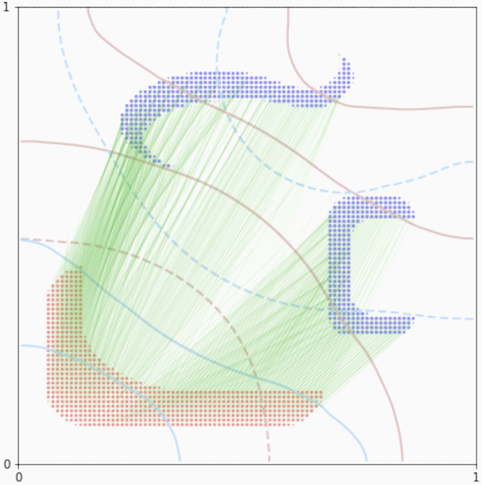
Key dates for discrete optimal transport with N points:

- [Kan42]: **Dual** problem of Kantorovitch.
- [Kuh55]: **Hungarian** methods in $O(\mathrm{N}^3)$.
- [Ber79]: **Auction** algorithm in $O(\mathrm{N}^2)$.
- [KY94]: **SoftAssign** = Sinkhorn + simulated annealing, in $O(\mathrm{N}^2)$.
- [GRL$^+$98, CR00]: **Robust Point Matching** = Sinkhorn as a loss.
- [Cut13]: Start of the **GPU era.**
- [Mér11, Lév15, Sch19]: **multi-scale** solvers in $O(\mathrm{N} \log \mathrm{N})$.

- **Solution,** today: **Multiscale Sinkhorn algorithm, on the GPU**.

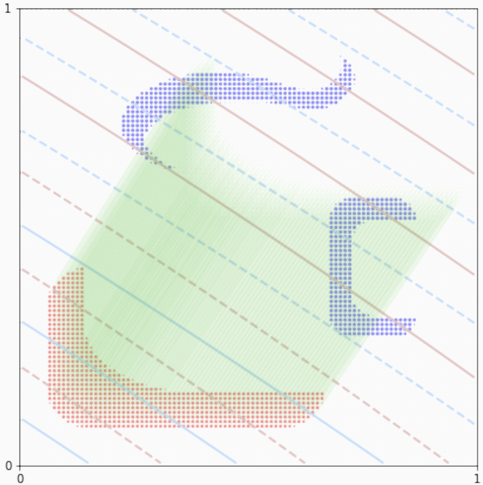
$\Longrightarrow$ Generalized **QuickSort** algorithm.

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$

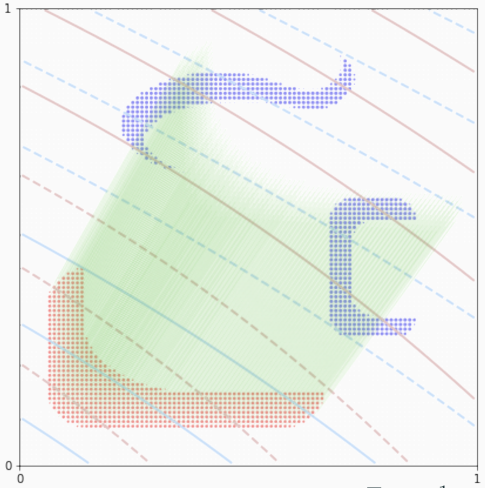


OT plan in 2D.

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$

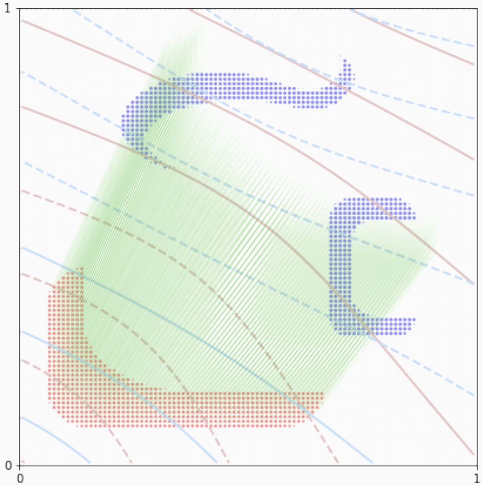


Iteration 0, blur $\sigma = \sqrt{\varepsilon} = 2^0$



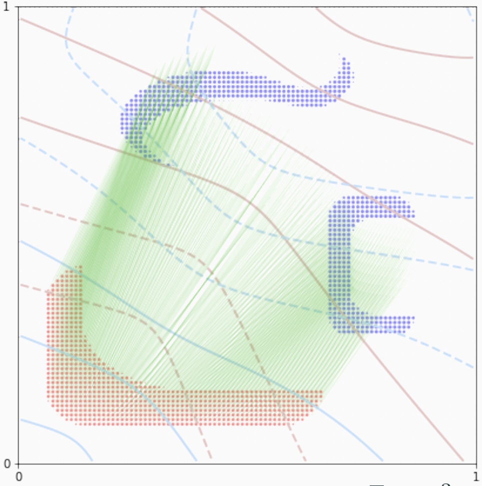
Iteration 1, blur $\sigma = \sqrt{\varepsilon} = 2^{-1}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



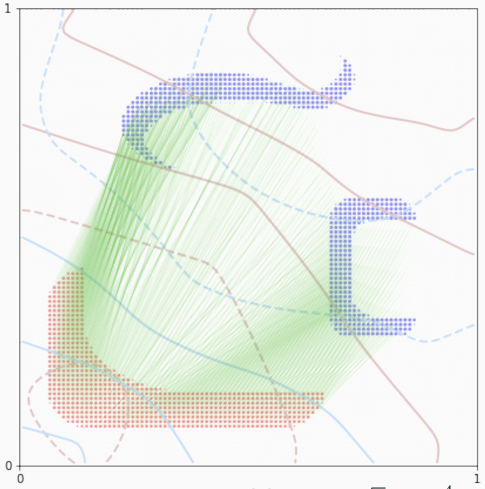
Iteration 2, blur $\sigma = \sqrt{\varepsilon} = 2^{-2}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathbf{OT}(\alpha, \beta)$



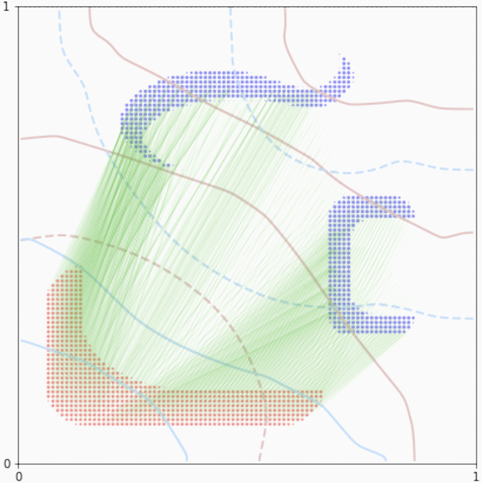
Iteration 3, blur $\sigma = \sqrt{\varepsilon} = 2^{-3}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



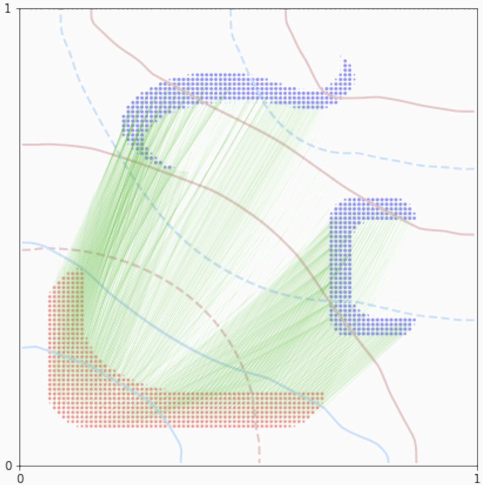
Iteration 4, blur $\sigma = \sqrt{\varepsilon} = 2^{-4}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\text{OT}(\alpha, \beta)$



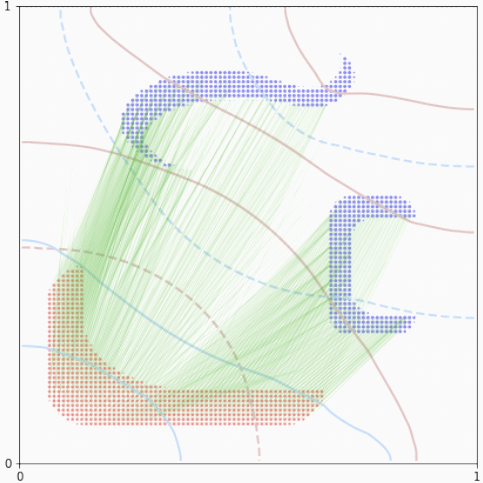
Iteration 5, blur $\sigma = \sqrt{\varepsilon} = 2^{-5}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathbf{OT}(\alpha, \beta)$



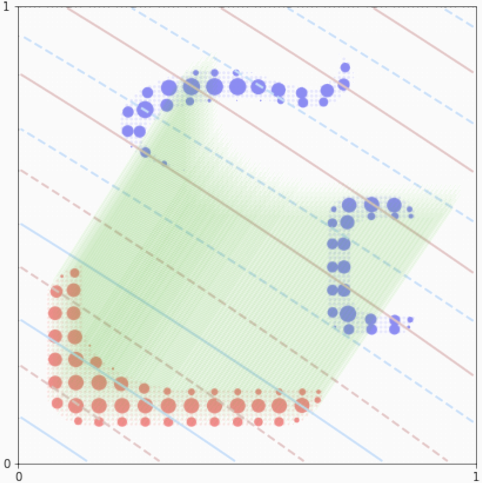
Iteration 6, blur $\sigma = \sqrt{\varepsilon} = 2^{-6}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



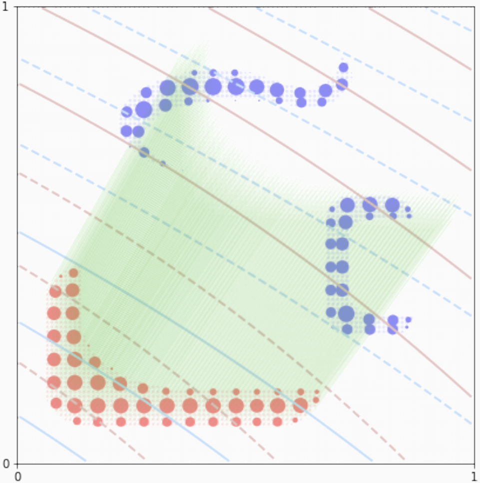
Iteration 7, blur $\sigma = \sqrt{\varepsilon} = .01$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha,\beta)$



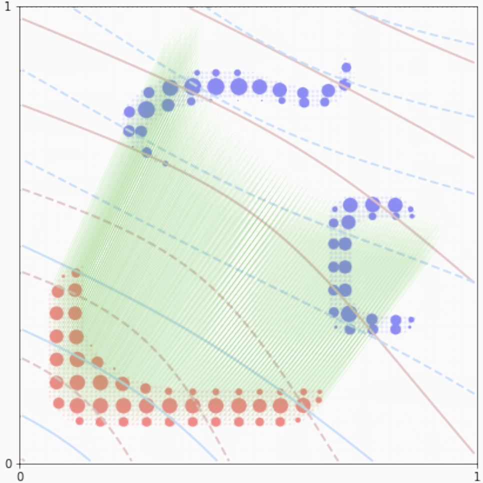
Iteration 0, blur $\sigma = 2^0$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\text{OT}(\alpha, \beta)$



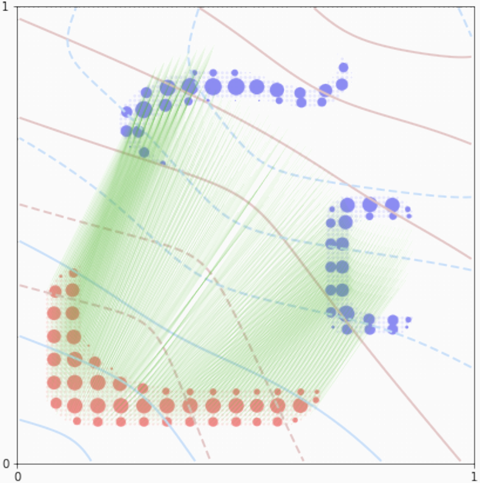
Iteration 1, blur $\sigma = 2^{-1}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



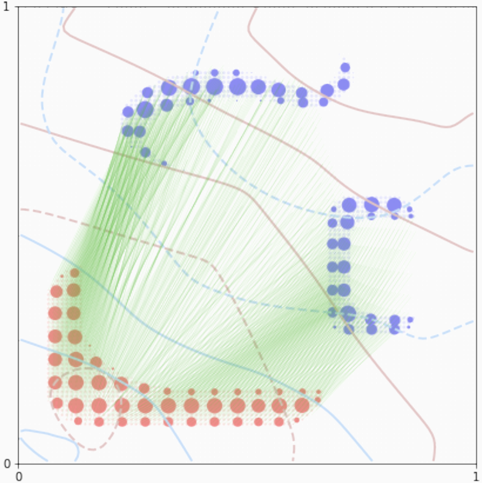
Iteration 2, blur $\sigma = 2^{-2}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha,\beta)$



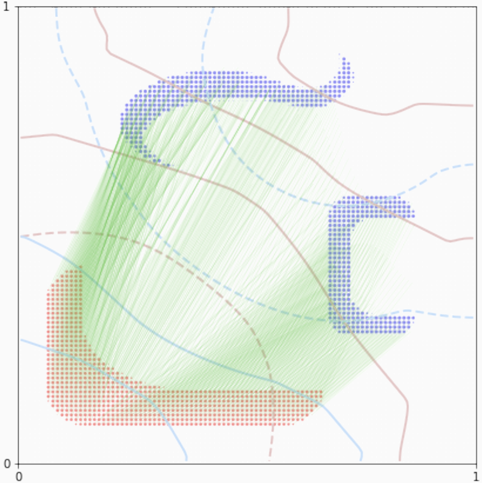
Iteration 3, blur $\sigma = 2^{-3}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\text{OT}(\alpha, \beta)$



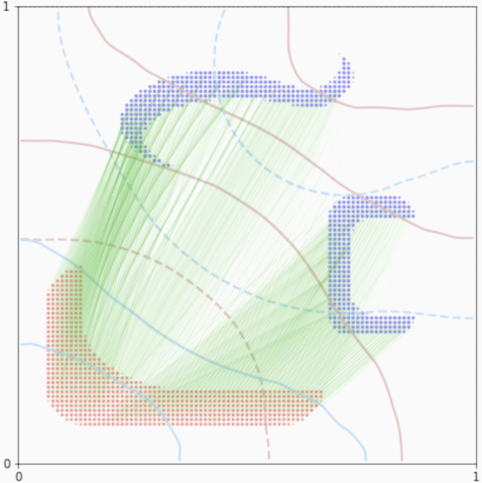
Iteration 4, blur $\sigma = 2^{-4}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



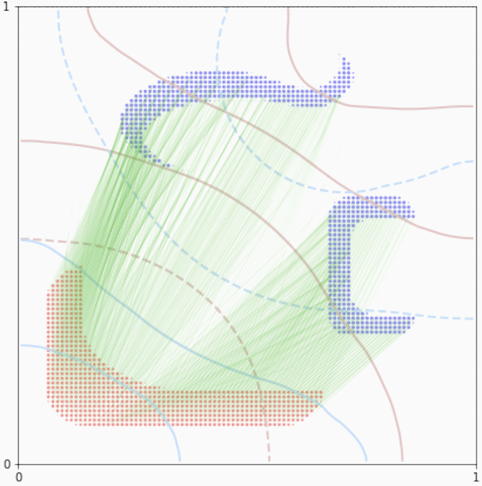
Iteration 5, blur $\sigma = 2^{-5}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathbf{OT}(\alpha, \beta)$



Iteration 6, blur $\sigma = 2^{-6}$

Visualizing F, G and the Brenier map $\nabla F(x_i) = -\frac{1}{\alpha_i}\partial_{x_i}\mathrm{OT}(\alpha, \beta)$



Iteration 7, blur $\sigma = .01$

Scaling up optimal transport to anatomical data

Progresses of the last decade add up to a $\times$**100** - $\times$**1000** acceleration:

$$\text{Sinkhorn GPU} \xrightarrow{\times 10} + \text{KeOps} \xrightarrow{\times 10} + \text{Annealing} \xrightarrow{\times 10} + \text{Multi-scale}$$

With a precision of 1%, on a modern gaming GPU:

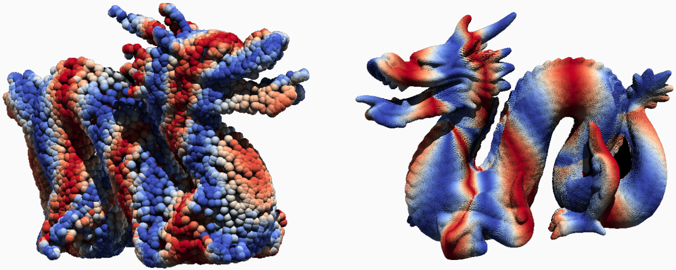
`pip install geomloss`

\+

modern GPU

(1 000 €)

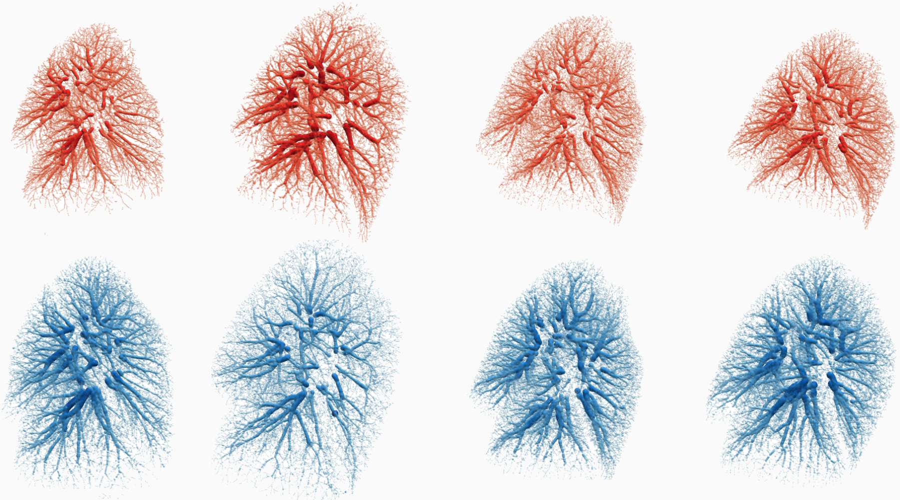
$\Longrightarrow$



10k points in 30-50ms

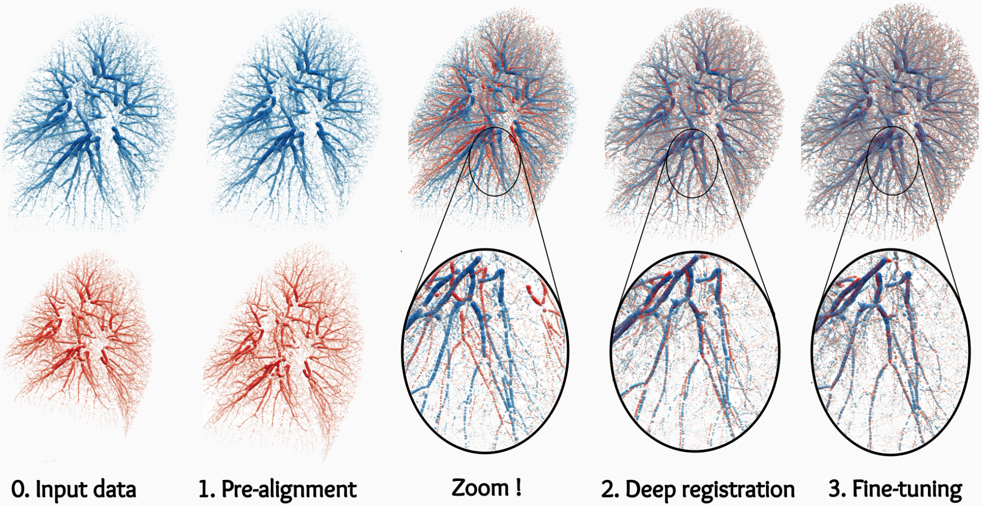
100k points in 100-200ms

A typical example in anatomy: lung registration "Exhale – Inhale"



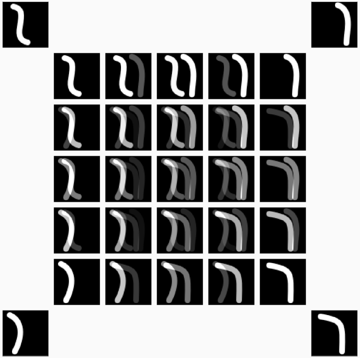
Complex deformations, high **resolution** (50k–300k points), high **accuracy** (< 1mm).

Three-steps registration



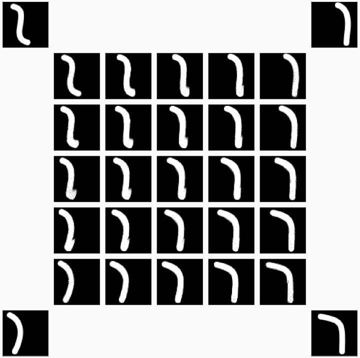
Wasserstein barycenters [AC11]

$$\text{Barycenter } A^* = \arg\min_{A} \sum_{i=1}^{4} \lambda_i \, \text{Loss}(A, B_i) \, .$$



Euclidean barycenters.

$$\text{Loss}(A, B) = \|A - B\|_{L^2}^2$$



Wasserstein barycenters.

$$\text{Loss}(A, B) = \text{OT}(A, B)$$

Incompressible particles

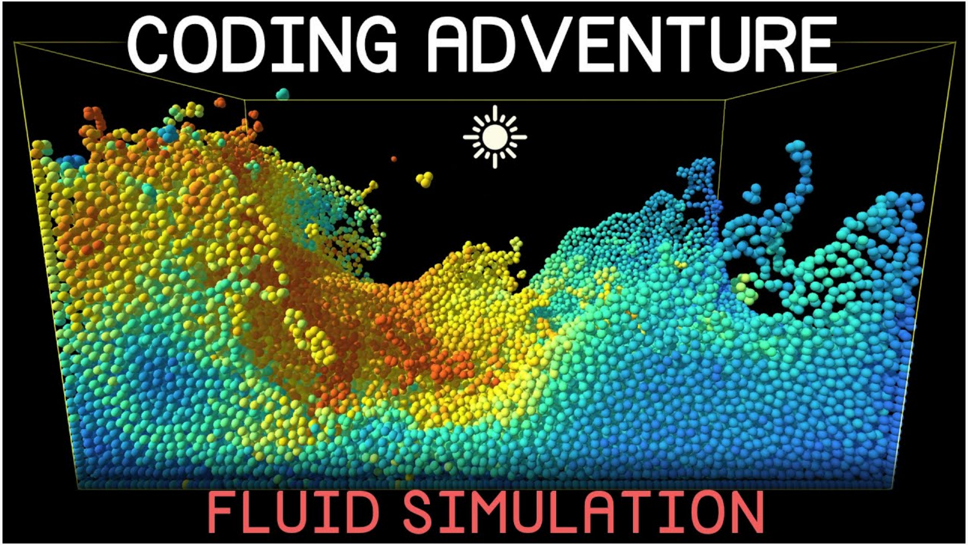
Two very talented colleagues

Maciej Buze
Lancaster University

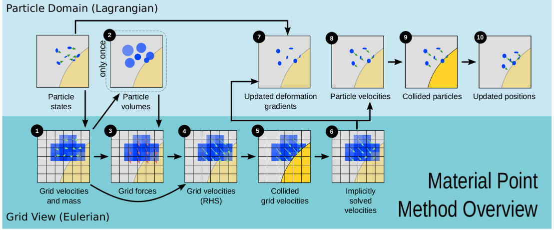
Antoine Diez
Kyoto University

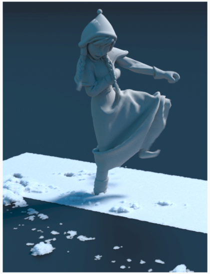
Original motivation: the N-body problem [Pri11]

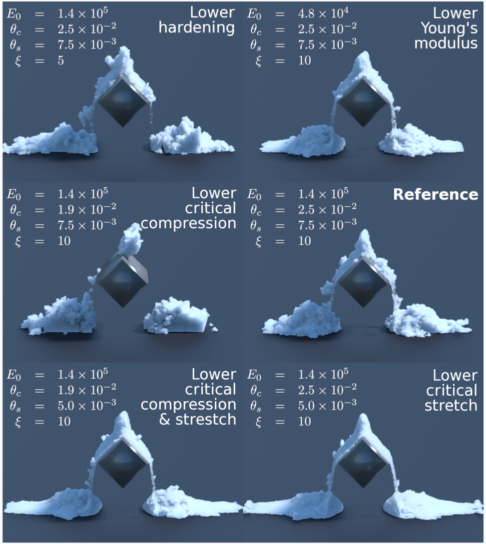
Coding a simple fluid simulation is now a matter of hours [Lag23]



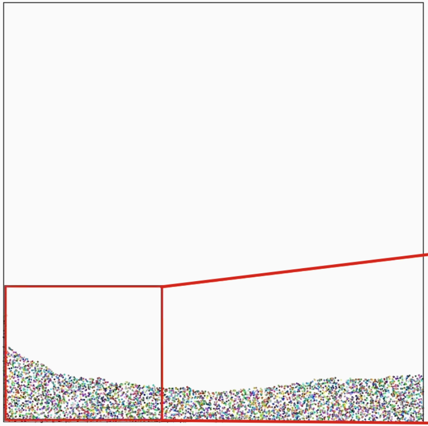
The material point method: Disney's Frozen [SSC+13]



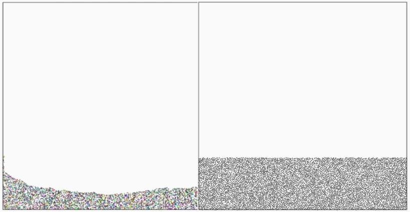




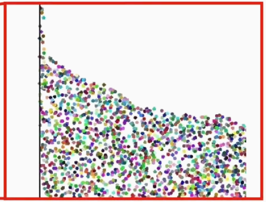
How can we enforce a volume preservation constraint? [QLDGJ22]



2D FLIP Simulation

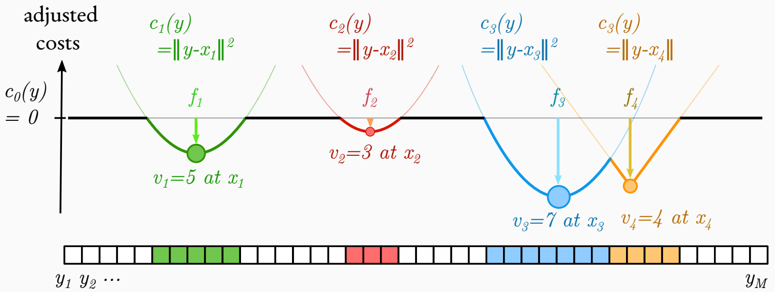


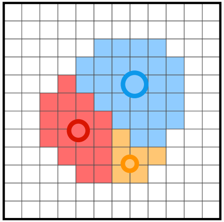
Volume loss!



Particle clumping and voids!

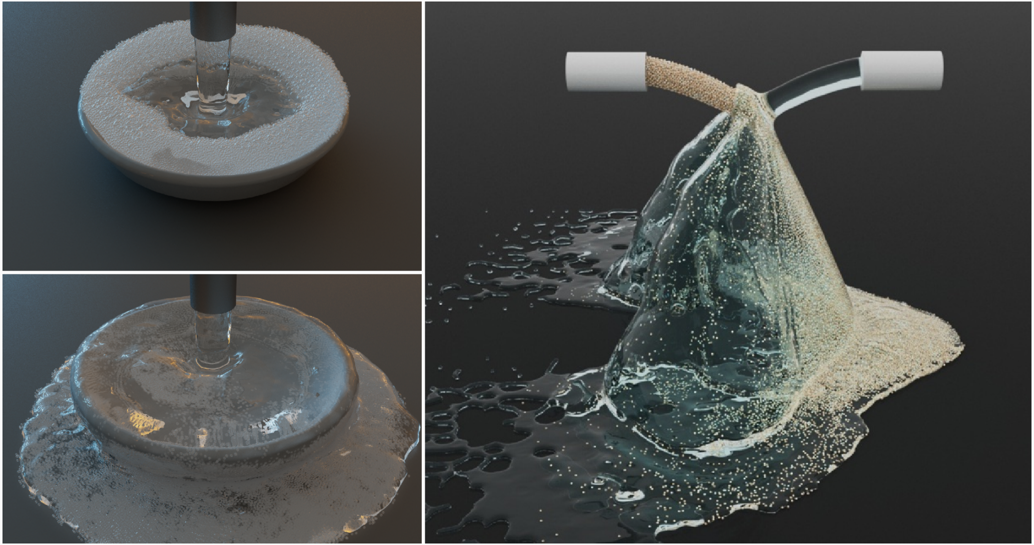
Use power diagrams i.e. semi-discrete optimal transport



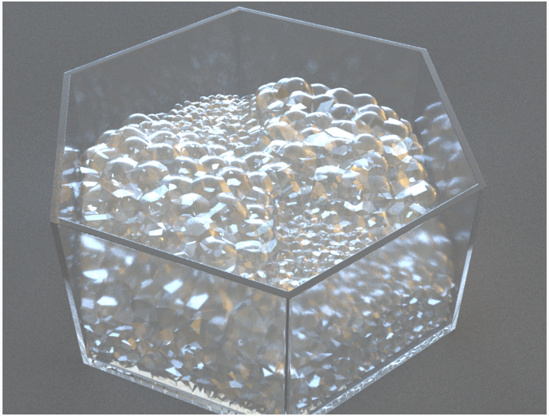


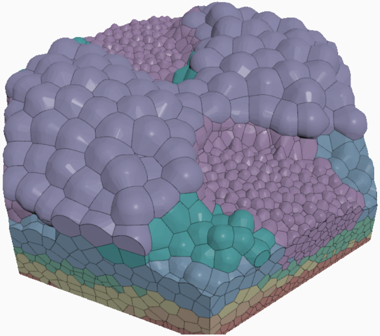
- The f_i's maximize the dual objective $\sum_{i=1}^N v_i f_i + \int_{y\in\Omega} \min_{i=0}^N [\, c_i(y) - f_i \,]\, \mathrm{d}y$.
- **Optimality** conditions $\iff$ $\mathrm{Vol}(\mathrm{Cell}_i) = v_i$.
- To **compute the cells**, the objective and its gradient:
    - If $c_i(y) = \|y - x_i\|^2$ for all cells, use a clever **grid-free** algorithm.
    - Otherwise, just use **KeOps**.

Power plastics [QLY+23]



Power plastics [QLY+23] – without the eye candy





Main numerical ingredients

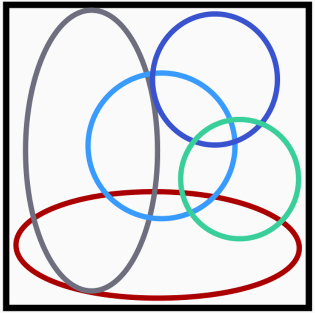
These simulations alternate between:

1. **Moving the particles** according to your favorite N-body model.
2. Computing Laguerre **cells** with the **correct volumes**:
   - (Multiscale) Sinkhorn for tolerance $> 5\%$.
   - (Quasi-)Newton for tolerance $< 1\%$.
3. **Correcting** the particle positions to enforce the volume-preservation constraint:
   - Jump to the centroid of the cell.
   - Or add a spring for smoother trajectories.

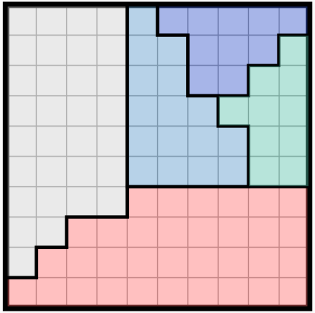
See e.g. Thomas Gallouët for a rigorous analysis with Mérigot, Lévy, etc.

But today: new applications with **custom cost functions** (thanks KeOps).

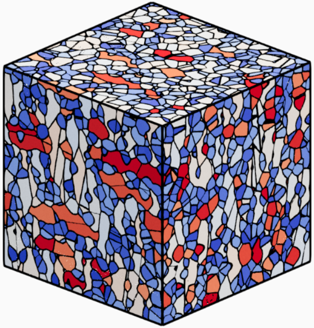
Anisotropic power diagrams let us model polycrystalline metals [BFR+24]



Ellipsoids.

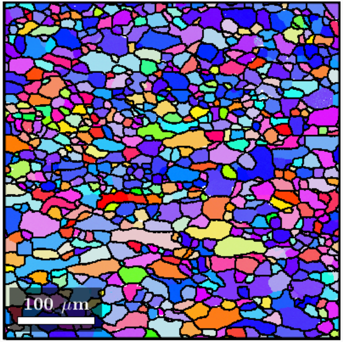


Pixel cells.

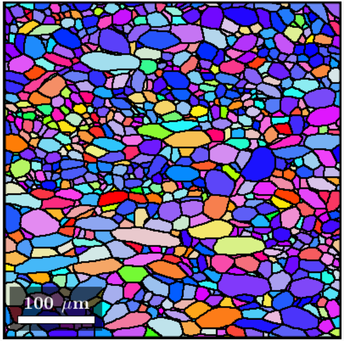


5,000 crystals in 3D.

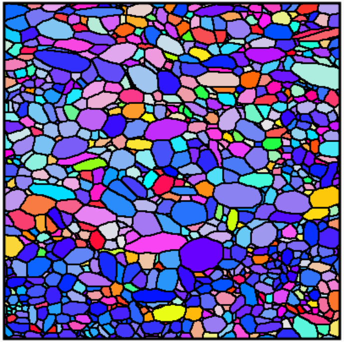
Fit to real EBSD scan of low-carbon steel [BFR+24]



Data from Tata steel.



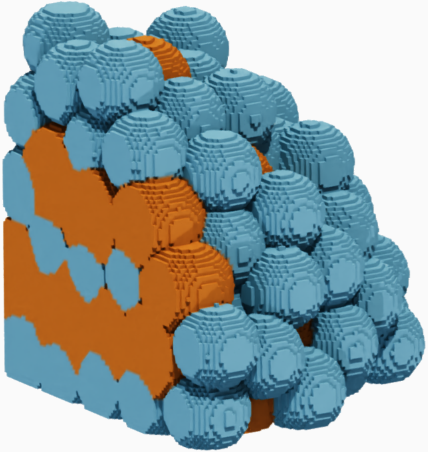
Our APD model.



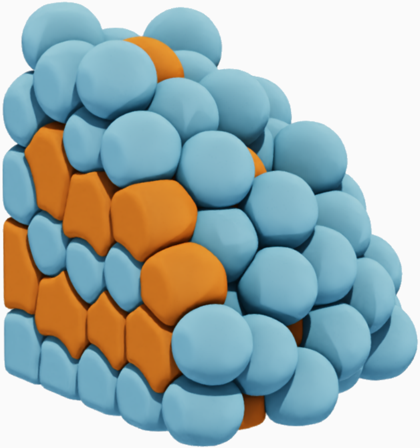
New synthetic image.

We can generate new, realistic 3D images with **prescribed properties** in seconds.

Change the cost function to simulate hard (blue) and soft (orange) cells [DF24]



The **raw** 100x100x100 pixel grid…

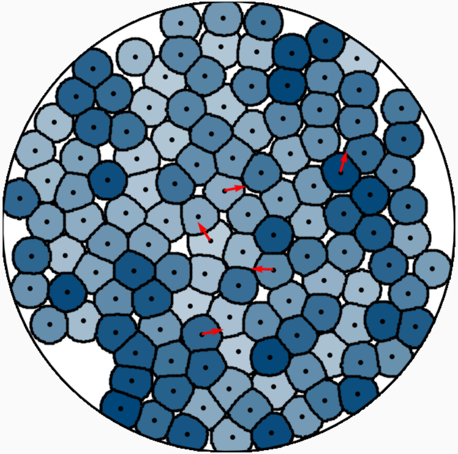


with some Hollywood **makeup**.

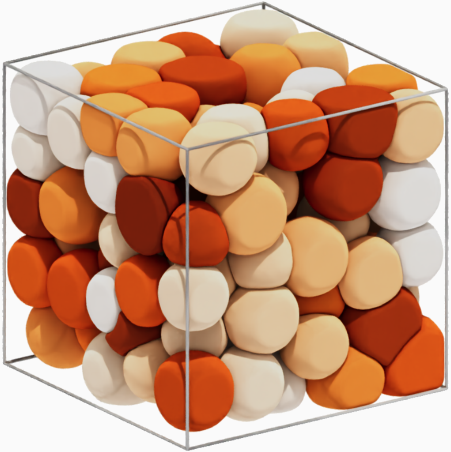
Let’s visit Antoine’s website

$\Longrightarrow$ https://iceshot.readthedocs.io $\Longleftarrow$

Run-and-tumble motion [DF24]

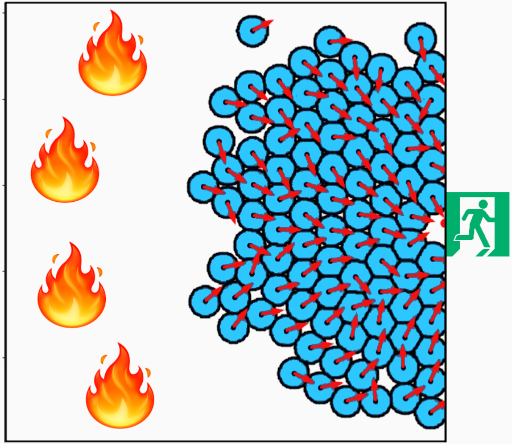


2D disk.

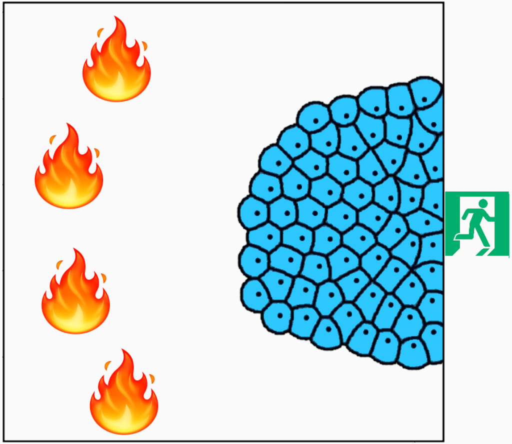


3D cube.

Fire alarm! [DF24]

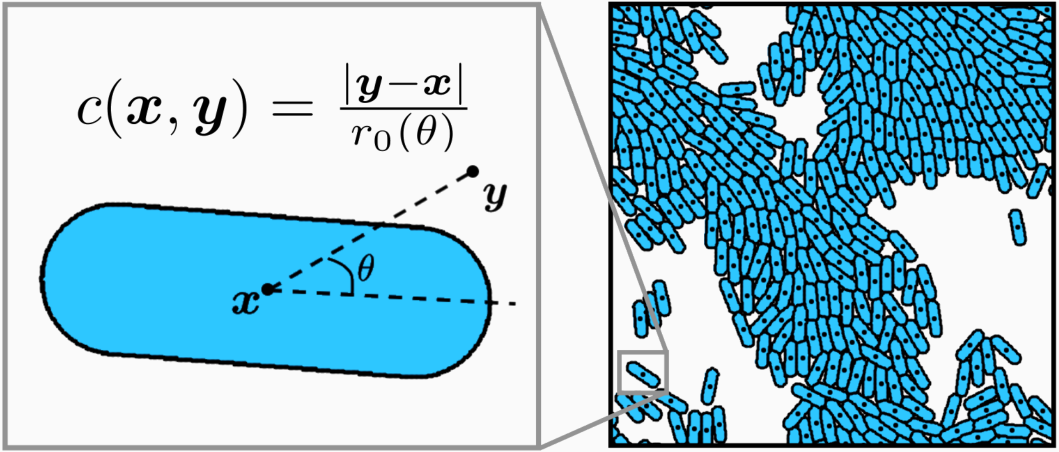


Hard particles **burn**.

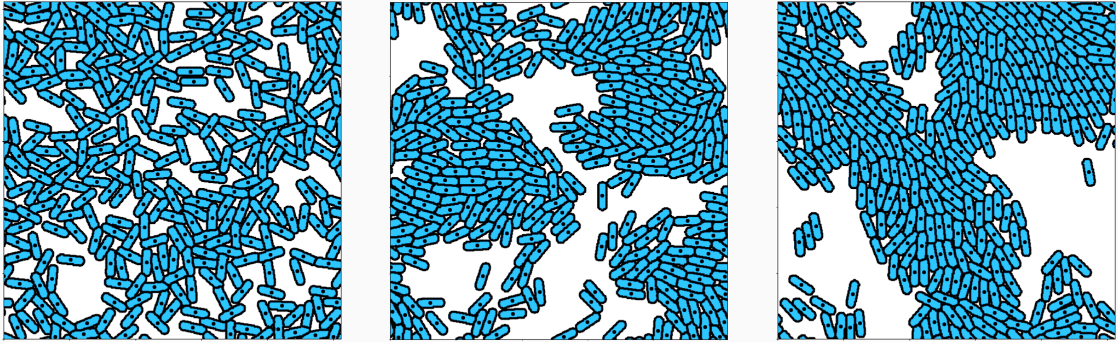


Soft particles **escape**.

Self-organizing swarms of blind, incompressible swimmers [DF24]



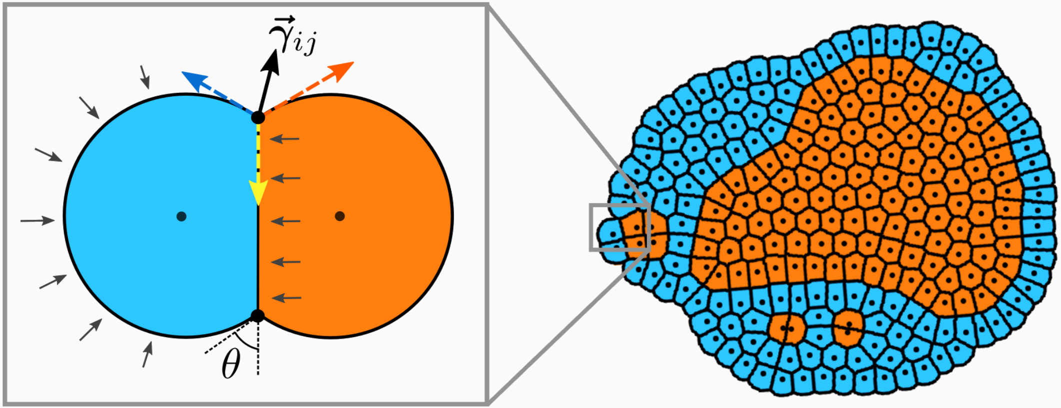
Self-organizing swarms of blind, incompressible swimmers [DF24]



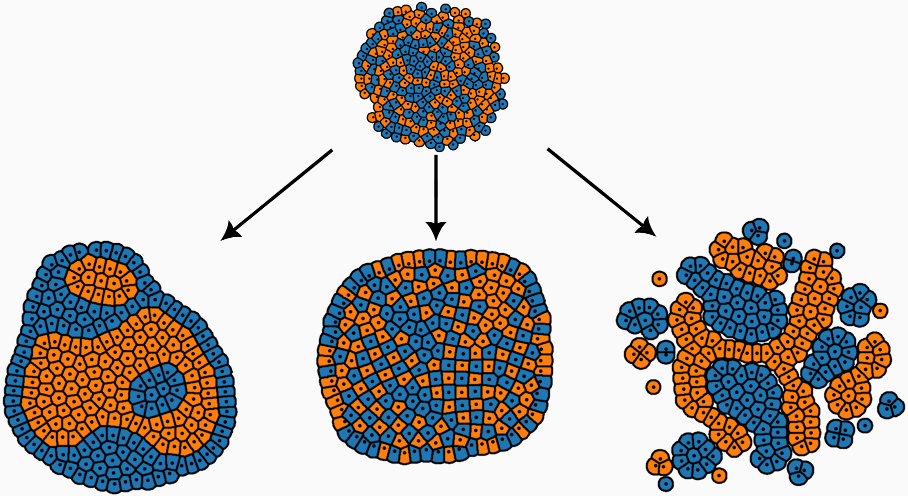
$t = 0$ $t = 4$ $t = 30$

Order emerges out of blind collisions and re-alignments.

Surface tension [DF24]



Surface tension [DF24] – playing with the energy parameters



Conclusion

Genuine team work

Benjamin Charlier

Joan Glaunès

Thibault Séjourné

F.-X. Vialard

Gabriel Peyré

Alain Trouvé

Marc Niethammer

Shen Zhengyang

Olga Mula

Hieu Do

Key points

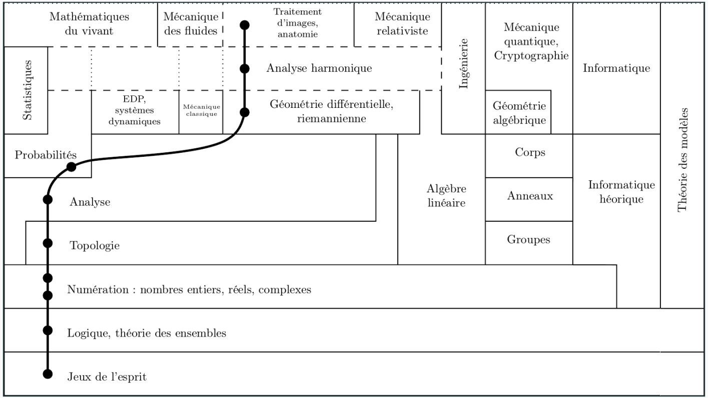
- Optimal Transport = volume preservation = **generalized sorting** :
  - $\longrightarrow$ Super-fast solvers on **simple domains**, especially 2D/3D spaces.
  - $\longrightarrow$ **Fundamental tool** at the intersection of geometry and statistics.

- The story is far from being over:
  - $\longrightarrow$ **OT** solvers $\simeq$ **linear** solvers: surprisingly interesting, but just a step.
  - $\longrightarrow$ Our main target: stronger metrics that preserve the **topology**.

- **Mathematics** is deeply relevant to **modern science and technology**:
  - $\longrightarrow$ Don't restrict yourselves to a narrow, elitist tunnel.
  - $\longrightarrow$ Exciting **career paths** and research directions.

An accessible textbook between fundamental and applied mathematics

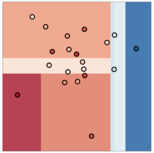
⟹ www.jeanfeydy.com/Teaching/culture_mathematique.pdf ⟸



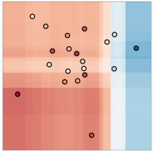
Lecture notes for my class of **“culture mathématique”** at the ENS.

Geometric data analysis

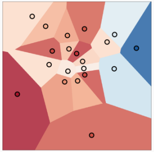
⟹ www.jeanfeydy.com/Teaching/ ⟸



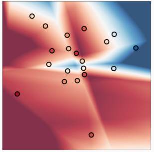
Decision tree.



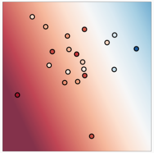
Random forest.



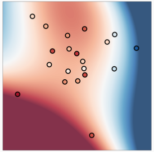
Nearest neighbors.



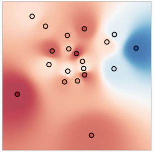
Neural network.



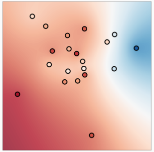
Linear.



Cubic.



Shepard.



Kernel.

A geometric perspective on data sciences, videos on YouTube.

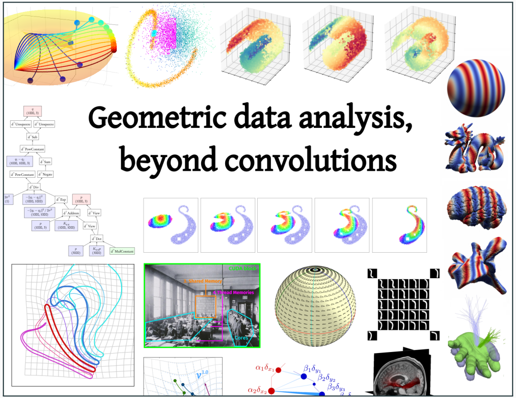
Some videos about modern 3D shape analysis

⟹ shape-analysis.github.io ⟸

Monthly seminar, videos on YouTube.

Documentation and references for the presentation

⟹ www.kernel-operations.io ⟸



www.jeanfeydy.com/geometric_data_analysis.pdf

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